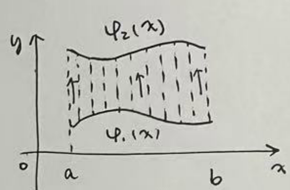
二重积分

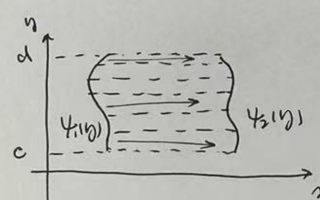
1. 定义 (Riemann和)

$$\iint_D f(x,y)\,dx\,dy := \lim_{\Delta\to 0}\sum_{i=1}^{n} f(x_i,y_i)\,\sigma_i$$

σ_i: 面积微元 / 分割.

2. 二重积分化为累次积分.





$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right] dx \qquad \iint_D f(x,y)\,dx\,dy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right] dy$$

or written as

$$= \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$$

$$= \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy.$$

其中 $I(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$ 是关于 x 的函数.
在积分时将 x 看作参数(常数).

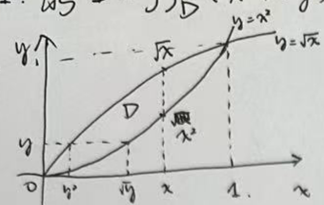
RMK. 1. 先画图, 画出积分区域.

2. 选择合适的积分顺序.

3. 计算 $\varphi_1(x)$, $\varphi_2(x)$ i.e. 积分区间.

4. 计算 $I(x)$ & $\int_a^b I(x)\,dx$.

例1. 求 $I=\iint_D (x^2+2y)\,dx\,dy$, D是 $y=x^2$, $y=\sqrt{x}$ 围成的区域.

解. 

(i) $I=\int_0^1\left[\int_{x^2}^{\sqrt{x}}(x^2+2y)\,dy\right]dx$

$=\int_0^1\left[I(x)\right]dx$.

$I(x)=(x^2y+y^2)\Big|_{x^2}^{\sqrt{x}} = x^{\frac{5}{2}}-x^4+x-x^4 = x^{\frac{5}{2}}+x-2x^4$

Then $I=\int_0^1\left[x^{\frac{5}{2}}+x-2x^4\right]dx = \frac{27}{70}$.

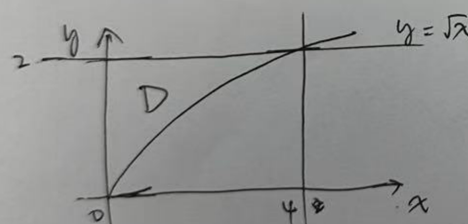
$=\frac{2}{7}+\frac{1}{2}-\frac{2}{5}=\frac{27}{70}$.

(ii) $I=\int_0^1\left[\int_{y^2}^{\sqrt{y}}(x^2+2y)\,dx\right]dy$

$I(y)=\frac{1}{3}x^3+2yx\Big|_{y^2}^{\sqrt{y}} = \cancel{\frac{1}{3}(y^{\frac{3}{2}}+2y^{\frac{3}{2}})} \; \frac{7}{3}y^{\frac{3}{2}}-\frac{1}{3}y^6-2y^3$

$I=\int_0^1\left(\frac{7}{3}y^{\frac{3}{2}}-\frac{1}{3}y^6-2y^3\right)dy = \frac{7}{3}\cdot\frac{2}{5}-\frac{1}{3}\cdot\frac{1}{7}-\frac{2}{4}=\frac{27}{70}$.

例2. $\iint_D \sin y^3\,dx\,dy$ D是 $y=\sqrt{x}$, $y=2$ 和 ~~$y=0$~~ $x=0$ 围成的区域.



$I=\int_0^4\left[\underline{\int_{\sqrt{x}}^2 \sin y^3\,dy}\right]dx$ 不知道原函数.

$I=\int_0^2\left[\int_0^{y^2}\sin y^3\,dx\right]dy$.

选择方案2. $\boxed{I(y)=}$

$I=\int_0^2 y^2\cdot\sin y^3\,dy \overset{t=y^3}{=} \frac{1}{3}\int_0^8 \sin t\,dt = -\frac{1}{3}\cos t\Big|_0^8$

$=\frac{1-\cos 8}{3}$.

3. 二重积分换元

$$\begin{cases} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{cases}$$

i.e. $(x,y) \longrightarrow (\xi,\eta)$. 换元.

$D \longrightarrow D'$

需要满足：

① 是一一映射（单、满）

② $x(\xi,\eta)$, $y(\xi,\eta)$ 有连续的偏导数.

③ Jacobi 行列式 $J = \frac{D(x,y)}{D(\xi,\eta)}$ 在 $(\xi,\eta) \in D'$ 处处非0.

则：$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(x(\xi,\eta), y(\xi,\eta))\,|J|\,d\xi\,d\eta$

J的绝对值.

$$dx\,dy = \left|\frac{D(x,y)}{D(\xi,\eta)}\right| d\xi\,d\eta.$$

极坐标换元：

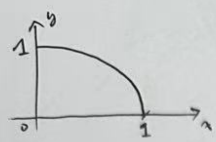
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad J = \frac{D(x,y)}{D(r,\theta)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r$$

$(r \geq 0,\ \theta \in [0, 2\pi))$

则 $\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

适用于积分区域与圆相关的情形. 或 积分函数含 (x^2+y^2).

例3. 求 $I=\iint_D (4-x^2-y^2)^{-\frac{1}{2}}\,dx\,dy$ D是单位圆盘在 $x^2+y^2\le 1$ 在 I 象限的部分.

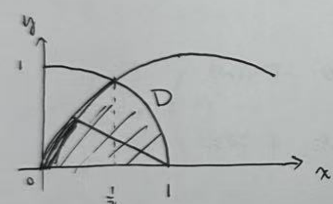
解. 

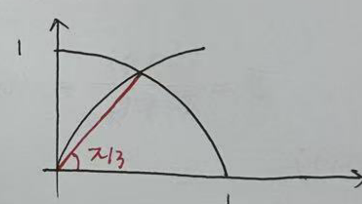
$D'=[0,1]\times[0,\frac{\pi}{2}]$

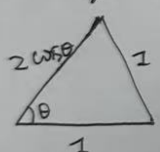
$$I=\iint_{D'}(4-r^2)^{-\frac{1}{2}}r\,dr\,d\theta$$
$$=\int_0^{\pi/2}\left[\int_0^1(4-r^2)^{-\frac{1}{2}}r\,dr\right]d\theta$$
$$=\int_0^{\pi/2}\left[-\sqrt{4-r^2}\Big|_0^1\right]d\theta$$
$$=\int_0^{\pi/2}(2-\sqrt{3})\,d\theta=\frac{\pi}{2}\cdot(2-\sqrt{3})$$

$$I=\iint_{D'}dr\,d\theta=\int_0^1(4-r^2)^{-\frac{1}{2}}r\left[\int_0^{\pi/2}1\,d\theta\right]dr=\frac{(2-\sqrt{3})}{2}\pi.$$

例4. $I=\iint_D(x^2+y^2)^{+\frac{1}{2}}\,dx\,dy$. D是 $x^2+y^2=1$ 与 $x^2-2x+y^2=0$ 的相交部分在 I 象限的部分.

解. 





$$D'\begin{cases}\theta\in[0,\frac{\pi}{3}]\text{时},\ r\in[0,1]\\ \theta\in(\frac{\pi}{3},\frac{\pi}{2}]\text{时},\ r\in[0,2\cos\theta]\end{cases}$$

$$I=\iint_{D'}r^2\,dr\,d\theta=\iint_{D_1'}+\iint_{D_2'}$$
$$=\int_0^{\pi/3}\left[\int_0^1 r^2\,dr\right]d\theta+\int_{\pi/3}^{\pi/2}\left[\int_0^{2\cos\theta}r^2\,dr\right]d\theta$$
$$=\int_0^{\pi/3}\frac{1}{3}\,d\theta+\int_{\pi/3}^{\pi/2}\frac{8}{3}\cos^3\theta\,d\theta$$
$$=\frac{\pi}{9}+\frac{8}{3}(\sin\theta-\sin^3\theta)\Big|_{\pi/3}^{\pi/2}=\frac{\pi}{9}+\frac{16}{9}-\sqrt{3}.$$

3. 可分离变量的二重积分.

$$I = \iint_{[a,b]\times[c,d]} f(x)\,g(y)\,dx\,dy$$

$$= \left(\int_{[a,b]} f(x)\,dx\right)\left(\int_{[c,d]} g(y)\,dy\right).$$

4. 高斯积分. $I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right)$$

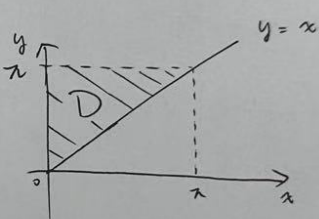
$$= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \iint_{[0,\infty)\times[0,2\pi)} e^{-r^2}\cdot r\,dr\,d\theta = \left(\int_0^{\infty} e^{-r^2} r\,dr\right)\cdot\left(\int_0^{2\pi} 1\,d\theta\right)$$

$$= \pi.$$

习题.

1. $I = \int_0^{\pi} dx \int_x^{\pi} \sin y / y \,dy$



解. $I = \iint_D \sin y / y \,dx\,dy$

$$= \int_0^{\pi} dy \int_0^{y} \frac{\sin y}{y}\,dx$$

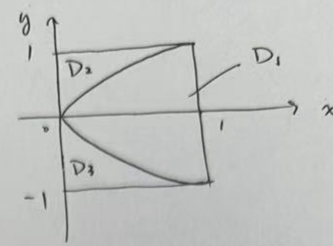
$$= \int_0^{\pi} \sin y\,dy = -\cos y\Big|_0^{\pi} = 2.$$

2. $I=\iint_D |x-y^2|\,dx\,dy$. $D=[0,1]\times[-1,1]$

· 讨论绝对值.

在 D_1 中: $|x-y^2| = x-y^2$

在 D_2, D_3 中: $|x-y^2| = y^2-x$.



$I=\iint_{D_1}+\iint_{D_2}+\iint_{D_3}$

$= I_1+I_2+I_3$

由对称性, $I_2=I_3=\int_0^1\left[\int_{\sqrt{x}}^1 (y^2-x)\,dy\right]dx$

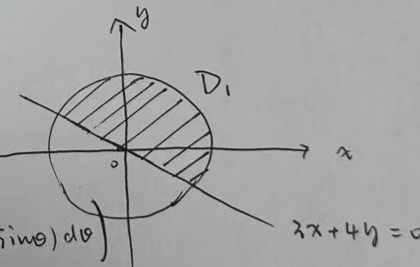
$=\int_0^1\left(\frac{1}{3}+\frac{2}{3}x^{\frac{3}{2}}-x\right)dx = \frac{1}{10}$

$I_1=\int_0^1\left[\int_{-\sqrt{x}}^{\sqrt{x}}(x-y^2)\,dy\right]dx=\int_0^1\frac{4}{3}x^{\frac{3}{2}}\,dx=\frac{8}{15}$

$I=I_1+I_2+I_3=\frac{11}{15}$.

3. $I=\iint_D |3x+4y|\,dx\,dy$ D是单位圆.

由对称性, $I=2\iint_{D_1} = 2I_1$



$I_1=\iint_{[0,1]\times[-\arctan\frac{3}{4},\,\pi-\arctan\frac{3}{4}]} r^2(3\cos\theta+4\sin\theta)\,dr\,d\theta$

$=\left(\int_0^1 r^2\,dr\right)\left(\int_{-\arctan\frac{3}{4}}^{\pi-\arctan\frac{3}{4}}(3\cos\theta+4\sin\theta)\,d\theta\right)$

$=\frac{1}{3}\cdot\left[(3\sin\theta-4\cos\theta)\Big|_{-\arctan\frac{3}{4}}^{\pi-\arctan\frac{3}{4}}\right]=\frac{10}{3}$.

$I=\frac{20}{3}$.

三重积分. 03/04/2025 孔舒阳

(I) Framework and definition of triple integral (n-multiple integral)

① Similar to double integral both in terms of def & tech. but caculation is more compliacted.

② In many cases. we can use some properties such as symmetry to simplify the calculation

1.1. Definition.

Step 1: Discretize Ω by Ω_i. $i=1,\cdots,n$. with $|\Omega_i| \leq \eta$

Step 2: Pick any point $(x_i, y_i, z_i) \in \Omega_i$

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\,|\Omega_i| \quad (|\Omega_i| \text{ is the volume of } \Omega_i)$$

Step 3: Taking limit. Let $\eta \to 0$.

$$\iiint_{\Omega} f(x, y, z)\,dx\,dy\,dz = \lim_{\eta \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,|\Omega_i|.$$

(Recall the definition of Riemann integral on 1-d)

1.2. Calculating integral in 3-D rectangular coordinates

Method: From 3-d integral to iterated integral

① Fix (x,y), integrate z.

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iint_{D}\left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right) dx\,dy.$$

② Fix z, integrate (x,y)

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \int_{z}\left(\iint_{D_z} f(x,y,z)\,dx\,dy\right) dz.$$

Practical steps:

Step 1: draw the area.

Step 2: choose method. ① or ②

Step 3: integrate according to ① or ②.

Some principles to choose ① or ②:

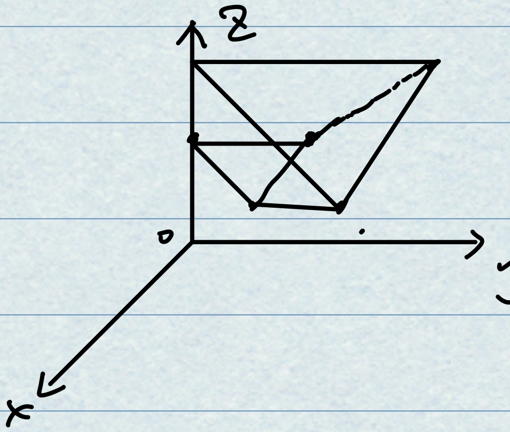
1. The form of integral area (try to avoid too many discussion)

Ex 1: $I = \iiint_{\Omega} (y^2+z^2)^{-1}\,dx\,dy\,dz$

Ω is formed by $\begin{cases} (0,0,1),\ (0,1,1),\ (1,1,1) \\ (0,0,2),\ (0,2,2),\ (2,2,2) \end{cases}$

Proof:

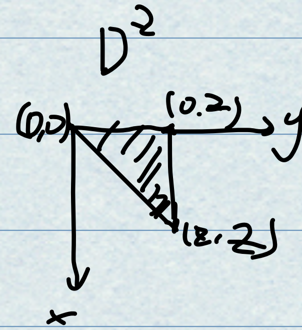
① Draw the area



parallel to xoy plane.

② fix z. $D^z(x,y)$ is the triangle by $(0,0,z)$ $(0,z,z)$ (z,z,z)

③ $$\int_1^2 \left(\iint_{D^z} (y^2+z^2)^{-1}\, dx\, dy \right) dz$$



$$\Rightarrow \iint_{D^z} (y^2+z^2)^{-1}\, dx\, dy$$

$$= \int_0^z \left(\int_0^y (y^2+z^2)^{-1}\, dx \right) dy = \int_0^z y(y^2+z^2)^{-1}\, dy$$

$$= \int_0^z (y^2+z^2)^{-1}\, d\left(\frac{y^2+z^2}{2}\right) = \frac{1}{2}\log(y^2+z^2)\Big|_0^z$$

$$= \frac{1}{2}\log(2z^2) - \log(z^2) = \frac{\log 2}{2}$$

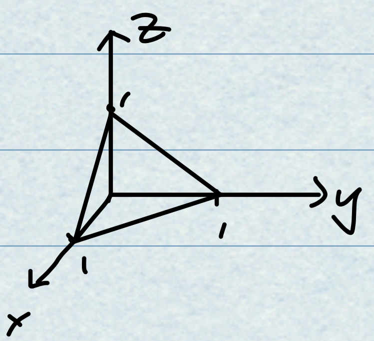
$$\Rightarrow I = \int_1^2 \frac{\log 2}{2}\, dz = \frac{\log 2}{2}$$

$\square$

2. The order of integral should take the closed form of the integral into account.

Ex 2. $I = \iiint_\Omega (1-y) e^{-(1-y-z)^2} \, dx\,dy\,dz$, where Ω is formed by $\begin{cases} x+y+z=1 \\ x=0 \\ y=0 \\ z=0 \end{cases}$

① Draw.



② Observation: we can't integrate y or z first due to the Gaussian integral.

$\Rightarrow$ integrate x first.

$$I = \iint_{D(y,z)} (1-y-z)(1-y) e^{-(1-y-z)^2} \, dy\,dz$$

$$= \int_0^1 \int_0^{1-y} (1-y-z)(1-y) e^{-(1-y-z)^2} \, dz \, dy$$

$$= \int_0^1 (1-y) \int_0^{1-y} \frac{1}{-2} e^{-(1-y-z)^2} \, d\left[(1-y-z)^2\right] \, dy$$

$$= \int_0^1 (1-y) \left(\frac{1}{2} e^{-(1-y-z)^2} \Big|_0^{1-y} \right) dy$$

$$= \int_0^1 (1-y)\frac{1}{2}\left(1 - e^{-(1-y)^2}\right) dy$$

$$= -\frac{1}{4}\left((1-y)^2 + e^{-(1-y)^2}\right)\Big|_0^1 = -\frac{1}{4}\left(1 - 1 - e^{-1}\right)$$

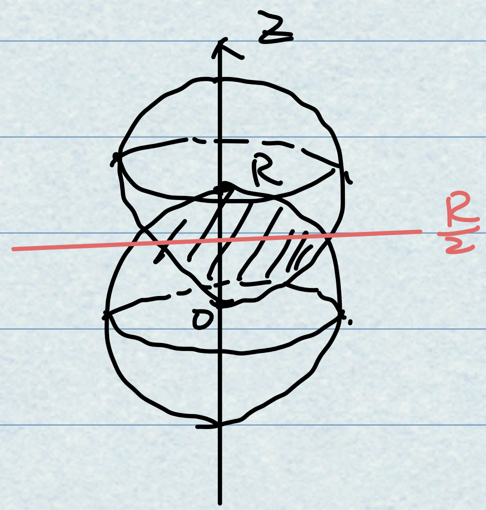
$$= \frac{1}{4e}$$

3. Some special functions. only depend on one of the x, y, z variables. If the function only depends on z, then integrate x, y first.

Ex3 $\iiint_\Omega z^2\, dx\,dy\,dz$, Ω is formed by.

$$\Omega := \begin{cases} x^2+y^2+z^2 \le R^2 \\ x^2+y^2+(z-R)^2 \le R^2 \end{cases}$$

① Draw.



details refer to Yantong Xie's notes.

Calculating by symmetry. (first consideration)

Ex 4 :

$$I = \iiint_\Omega (x+1)(y+1)\, dx\,dy\,dz$$

$$\Omega: \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$$

proof:

$$(x+1)(y+1) = xy + y + x + 1$$

$I = I_1 + I_2 + I_3 + I_4$, where

$$I_1 = \iiint_\Omega xy\,dx\,dy\,dz$$

$$I_2 = \iiint_\Omega y\,dx\,dy\,dz$$

$$I_3 = \iiint_\Omega x\,dx\,dy\,dz$$

$$I_4 = \iiint_\Omega dx\,dy\,dz$$

By symmetry, $I_1 = I_2 = I_3 = 0$

$$I_4 = V = \frac{4}{3}\pi abc$$

1.3. change of variables.

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$

three conditions:

① bijection : $\Omega' \to \Omega$

② $x(\,)$, $y(\,)$, $z(\,)$ continuous partial derivatives

③ Jacobian: $J \triangleq \dfrac{D(x,y,z)}{D(u,v,w)} \neq 0$

Then,

$$\iiint_{\Omega} f(x,y,z)\,dV = \iiint_{\Omega'} f(x(u,v,w), y(u,v,w), z(u,v,w))\,|J|\,du\,dv\,dw.$$

Two important coordinates:

① Cylindrical coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$\Rightarrow J = r$

$$\iiint_{\Omega} f(x,y,z)\,dV = \iiint_{\Omega} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

Ex 5 $I = \iiint_{\Omega} (x^2+y^2)^{\frac{1}{2}}\,dx\,dy\,dz$.

$$\Omega = \begin{cases} x^2+y^2 = 9 \\ x^2+y^2 = 16 \\ z = 0 \\ z = \sqrt{x^2+y^2} \end{cases}$$

Pf: $I = \iiint_{\Omega} r^2\,dr\,d\theta\,dz$

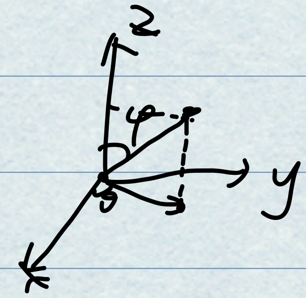
$$\Omega = \begin{cases} 3 \le r \le 4 \\ 0 \le z \le r \end{cases}$$

$$I = \int_0^{2\pi}\int_3^4 \left(\int_0^r r^2\,dz\right) dr\,d\theta = \int_0^{2\pi}\int_3^4 r^3\,dr\,d\theta$$

$$= 2\pi \cdot \frac{r^4}{4}\Big|_3^4 = \frac{175}{2}\pi$$

② Spherical coordinates

$$\begin{cases} z = \rho\cos\varphi \\ x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \end{cases}$$



$$J = \rho^2 \sin(\varphi)$$

$$\Rightarrow \iiint_\Omega f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_{\Omega'} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Ex 6. $\iiint_D y^2\,dx\,dy\,dz \qquad D: x^2+y^2+z^2 \le 2z$

pf:

Obs: $D \mapsto D'$ where $D': x^2+y^2+z^2 \le 1$

the integral is invariant

$$\Rightarrow \iiint_D \rho^2\sin^2\varphi\sin^2\theta\;\rho^2\sin\varphi\;d\rho\,d\varphi\,d\theta$$

$$= \iiint_D \rho^4 \sin^3\varphi\,\sin^2\theta\,d\rho\,d\varphi\,d\theta$$

$$= \left(\int_0^1 \rho^4 \, d\rho\right)\left(\int_0^{\pi} \sin^3\varphi \, d\varphi\right)\cdot\left(\int_0^{2\pi} \sin^2\theta \, d\theta\right)$$

$$= \frac{1}{5}\cdot\frac{4}{3}\cdot\pi = \frac{4}{15}\pi$$

$\left(\text{Recall the integral } I = \int_0^{\frac{\pi}{2}} \sin^n(\theta)\, d\theta\right)$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n(\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-1}(\theta)\, d(-\cos\theta)$$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta \,(n-1)\sin^{n-2}(\theta)\, d\theta$$

$$= (n-1)\, I_{n-2} - (n-1)\, I_n$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}$$

$$\Rightarrow I_n = \begin{cases} \dfrac{(n-1)!!}{n!!} & n \text{ odd} \\[2ex] \dfrac{(n-1)!!}{n!!}\cdot\dfrac{\pi}{2} & n \text{ even} \end{cases}$$

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Exercise 1:

$$I=\iiint_\Omega (z+z^2)\,dV \qquad \Omega:\ x^2+y^2+z^2\le 1$$

Pf: (Use symmetry)

$$I=\iiint_\Omega z^2\,dV$$

symmetry.

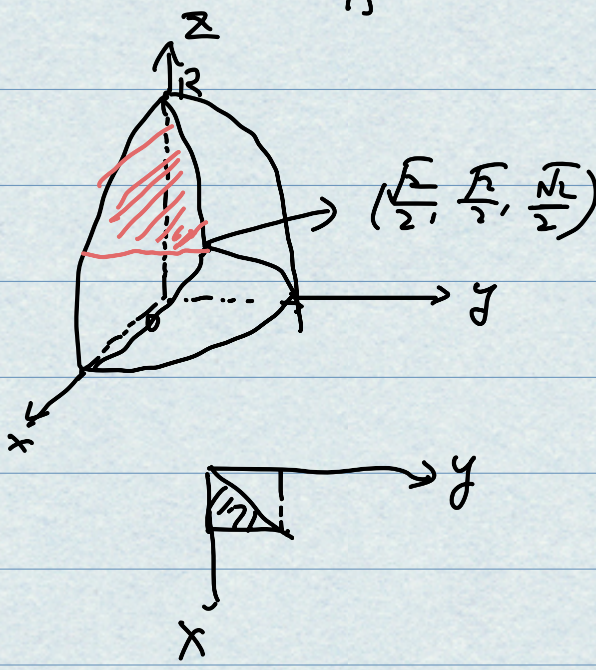
$$\boxed{=}\ \frac{1}{3}\iiint_\Omega (x^2+y^2+z^2)\,dV$$

$$=\frac{1}{3}\int_0^\pi\int_0^{2\pi}\int_0^1 \rho^2\cdot\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$=\frac{1}{3}\cdot 2\pi\cdot\frac{1}{5}\cdot\int_0^\pi \sin\varphi\,d\varphi=\frac{4\pi}{15}.$$

$\square$

Exercise 2. (☆)



Calculate the surface measure

$$\begin{cases} x^2+y^2=R^2\\ x^2+y^2=R^2\\ y^2+z^2=R^2\end{cases}$$

$S=6S'$

$$z=\sqrt{R^2-x^2}=:f_2(x,y)$$

$$\frac{\partial z}{\partial x}=\frac{(-x)}{\sqrt{R^2-x^2}}$$

$$S'=\iint_{D_{xy}}\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx\,dy=\iint_{D_{xy}}R\cdot\frac{1}{\sqrt{R^2-x^2}}\,dx\,dy$$

$$= R\int_0^{\frac{\sqrt{2}}{2}R} \frac{x}{\sqrt{R^2-x^2}}\,dx = -\frac{R}{2}\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{R^2-x^2}}\,d(R^2-x^2)$$

$$= -\frac{R}{2}\cdot 2\,(R^2-x^2)^{\frac{1}{2}}\Big|_0^{\frac{\sqrt{2}}{2}R} = -\left(R\left(R^2-\frac{R^2}{2}\right)^{\frac{1}{2}} - R^2\right)$$

$$= R^2\cdot\left(1-\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow S = 6S' = 6R^2\left(1-\frac{\sqrt{2}}{2}\right)$$

<u>Ex 3</u>: $V = 6V'$

$$V' = \iiint_\Omega 1\,dx\,dy\,dz \qquad \Omega: \begin{cases} x^2+y^2 \le R^2 \\ x \ge y \\ z \le y \end{cases}$$

$$V' = \int_0^{\frac{\pi}{4}}\int_0^R\int_0^{r\sin\theta} r\,dz\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}}\int_0^R r^2\sin\theta\,dr\,d\theta$$

$$= \frac{R^3}{3}\cdot(-\cos\theta)\Big|_0^{\frac{\pi}{4}} = \frac{R^3}{3}\left(1-\frac{\sqrt{2}}{2}\right)$$

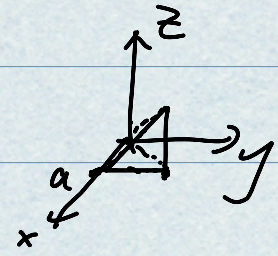
$$\Rightarrow V = 2R^3\left(1-\frac{\sqrt{2}}{2}\right)$$

Exercise 4.

$$I = \int_0^a dx \int_0^x dy \int_0^y f(z)\,dz$$

$$= \iiint_\Omega f(z)\,dx\,dy\,dz$$

$$= \int_0^a f(z) \underbrace{\tfrac{1}{2}(a-z)^2}_{\text{area of triangle.}} dz$$



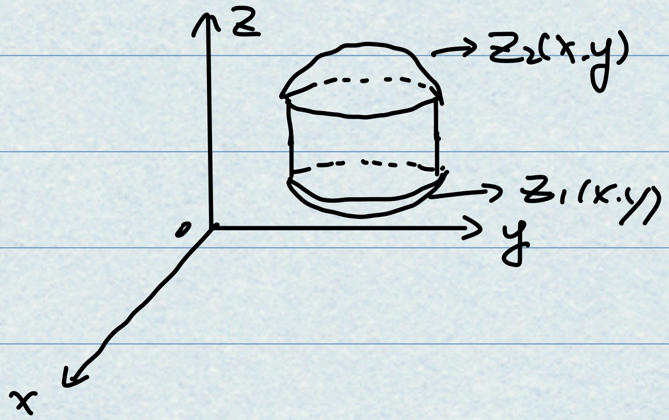
知识点回顾.

① 区域 Ω 的体积 V.

$$V = \iiint_\Omega 1\,dx\,dy\,dz$$

常用方法.

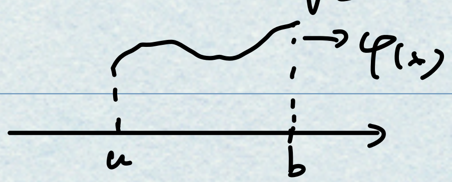
$$V = \iiint_D \left(z_2(x,y) - z_1(x,y)\right) dx\,dy$$



② 旋转体的体积

设 V 由 $y = \varphi(x)$ $a \le x \le b$ 绕 x 轴旋转的体积

$$V = \pi \int_a^b \varphi^2(x)\,dx$$



③ 曲面面积

$$z=f(x,y),\quad (x,y)\in D$$

$$S=\iint_D \sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,dxdy$$

④ 质量

$$M=\iint_D \rho(x,y)\,dxdy \qquad \text{平面}$$

$$M=\iiint_D \rho(x,y,z)\,dxdydz \qquad \text{三维空间.}$$

⑤ 质心

$$x_0=\frac{\iiint_\Omega x\rho(x,y,z)\,dV}{M}$$

y_0, z_0 类似.

二重积分, 三重积分 头等大事是计算: 算对!

<u>Exercise 5</u>:

求第一卦限中的部分椭球体

$$\Omega: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1 \quad (x\geq 0\ \ y\geq 0,\ z\geq 0)$$

的重心坐标. (设密度 $\rho=1$)

解:

Step1: 先算体积,

$$M=\iiint_\Omega dxdydz=\int_0^c\left(\iint_{D(z)}dxdy\right)dz$$

其中 $D(z)$: $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1-\frac{z^2}{c^2}$

$$\Rightarrow S(D(z)) = \frac{1}{4}\pi ab\left(1-\frac{z^2}{c^2}\right)$$

$$\Rightarrow M = \int_0^c \frac{1}{4}\pi ab\left(1-\frac{z^2}{c^2}\right)dz$$

$$= \frac{\pi ab}{4}\left(z-\frac{z^3}{3c^2}\right)\Big|_0^c = \frac{\pi ab}{4}\,\frac{2c}{3} = \frac{\pi abc}{6}$$

Step 2:

$$\iiint_V z\,dx\,dy\,dz$$

$$= \int_0^c z\,dz\cdot\left(\iint_{D(z)} dx\,dy\right)$$

$$= \int_0^c z\,\frac{1}{4}\pi ab\left(1-\frac{z^2}{c^2}\right)dz$$

$$= \frac{\pi ab}{4}\int_0^c\left(z-\frac{z^3}{c^2}\right)dz$$

$$= \frac{\pi ab}{4}\left(\frac{z^2}{2}-\frac{z^4}{4c^2}\right)\Big|_0^c = \frac{\pi ab}{4}\cdot\frac{c^2}{4} = \frac{\pi abc^2}{16}.$$

$$\Rightarrow z_0 = \frac{\frac{\pi abc^2}{16}}{\frac{\pi abc}{6}} = \frac{3}{8}c$$

同理: $x_0=\frac{3}{8}a \quad y_0=\frac{3}{8}b$

□

Exercise 6 设 f 在 $[0,1]$ 上为正连续函数, 证明

记 $I = \int_0^1 f(x)\,dx \int_0^1 \frac{1}{f(x)}\,dx$

(i) $1 \le I \le \frac{m^2+M^2}{2mM}$

(ii) $I \le \frac{(m+M)^2}{4mM}$

Pf:

(i) $I = \int_0^1 \frac{1}{f(x)}\,dx \int_0^1 f(x)\,dx$

$$= \int_{[0,1]\times[0,1]} \frac{f(y)}{f(x)}\,dx\,dy$$

$$= \frac{1}{2}\int_{[0,1]\times[0,1]} \left(\frac{f(y)}{f(x)} + \frac{f(x)}{f(y)}\right) dx\,dy$$

令 $t = \frac{f(y)}{f(x)}$, $\frac{m}{M} \le t \le \frac{M}{m}$.

考虑 $f(t) = t + \frac{1}{t}$ $\quad t \in [\frac{m}{M}, \frac{M}{m}]$

$\Rightarrow \quad 1 \le I \le \frac{1}{2}\left(\frac{M}{m} + \frac{m}{M}\right) = \frac{m^2+M^2}{2mM}$ $\quad\square$

(ii)

$$I = \int_0^1 \frac{\sqrt{mM}}{f(x)}\,dx \int_0^1 \frac{f(x)}{\sqrt{mM}}\,dx$$

$$\le \frac{1}{4}\left[\int_0^1 \left(\frac{\sqrt{mM}}{f(x)} + \frac{f(x)}{\sqrt{mM}}\right) dx\right]^2$$

令 $t = \frac{\sqrt{mM}}{f(x)}$ $\quad \sqrt{\frac{m}{M}} \le t \le \sqrt{\frac{M}{m}}$

$\Rightarrow I \le \frac{1}{4}\left(\sqrt{\frac{m}{M}} + \sqrt{\frac{M}{m}}\right)^2 = \frac{(m+M)^2}{4mM}$ $\quad\square$

(Hölder 不等式)

Exercise 7, $f = f(x,y)$ on Ω.

令 $\|f\|_p = \left(\iint_\Omega |f(x,y)|^p \, dx\, dy\right)^{\frac{1}{p}}$ 设 $\|f\|_p < +\infty$

$\downarrow$

L^p norm of f

设 $p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$，则

$$\|uv\|_1 \le \|u\|_p \|v\|_q$$

Hint: $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ (Young 不等式)

Pf:

设 $a = \frac{|u|}{\|u\|_p}$ $\quad b = \frac{|v|}{\|v\|_q}$

$$\Rightarrow \frac{|u|\cdot|v|}{\|u\|_p \|v\|_q} \le \frac{1}{p}\frac{|u|^p}{\|u\|_p^p} + \frac{1}{q}\frac{|v|^q}{\|v\|_q^q}$$

$$\iint_\Omega \frac{|u||v|}{\|u\|_p\|v\|_q}\, dx\,dy$$

$$\le \frac{1}{p}\iint_\Omega \frac{|u|^p}{\|u\|_p^p}\, dx\,dy + \frac{1}{q}\iint_\Omega \frac{|v|^q}{\|v\|_q^q}\, dx\,dy$$

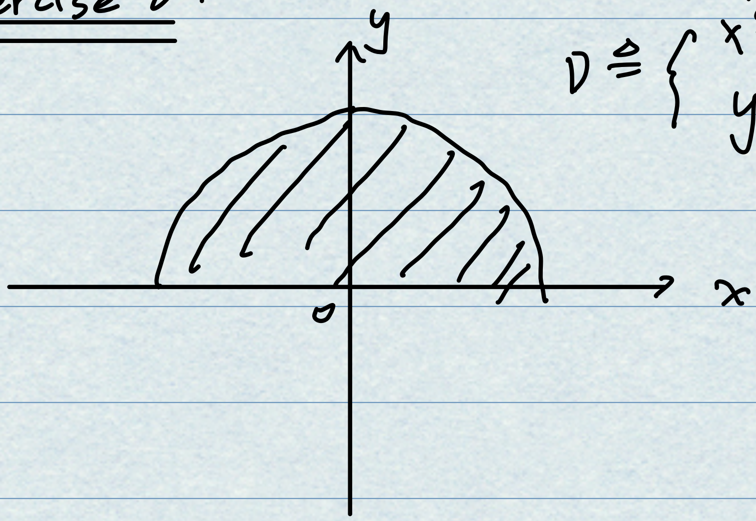
$$= \frac{1}{p}+\frac{1}{q} = 1$$

$\boxed{22}$

更一般地，有 $\sum_{i=1}^{m} \frac{1}{p_i} = 1$

$$\iint_\Omega u_1 u_2 \cdots u_m \, dx\,dy \le \|u_1\|_{p_1} \cdots \|u_m\|_{p_m}$$

Exercise 8:



$$D \triangleq \begin{cases} x^2+y^2 \le R^2 \\ y \ge 0 \end{cases}$$

求 D 的质心: (x_0, y_0)

Pf: 由对称性 $x_0 = 0$

① $S = \frac{1}{2}\pi R^2$

② $$\iint_D y \, dxdy = \int_0^\pi \int_0^R r\sin\theta \, r \, dr \, d\theta$$

$$= \frac{2R^3}{3}$$

$$y_0 = \frac{\iint_D y\,dxdy}{S} = \frac{2}{3}R^3 \cdot \frac{2}{\pi R^2} = \frac{4R}{3\pi}$$

□

曲线积分的定义

回顾：

定积分和重积分中，三要素为"积分区域"、"被积函数"和"积分微元"。曲线积分也分为三步骤：①分，②积，③取极限

第一型曲线积分：

① 分：将曲线分成若干小段 L_i

② 积：在每一小段取点 (x_i, y_i)，求和 $\sum f(x_i, y_i)\Delta s_i$，$\Delta s_i$ 是每个小段的长度

③ 取极限：不停加细曲线划分，使每个 Δs_i 趋于 0，和式极限定义为第一型曲线积分

$$\int_L f(x,y)\,ds = \lim_{\lambda\to 0} \sum f(x_i, y_i)\Delta s_i$$

其中 $\lambda = \max |\Delta s_i|$

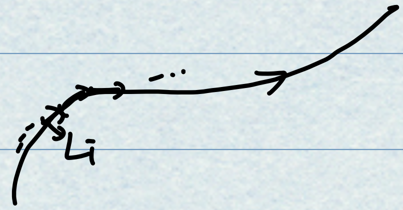
理解：求一条曲线 L 的质量

f 可看作是密度 $\rho(x,y)$

第二型曲线积分：

考虑向量值函数 $\vec{F}(x,y) = (P(x,y), Q(x,y))$，即 $\vec{F}$ 为一个从 (x,y) 平面到 $\mathbb{R}^2$ 的映射

① 分：将曲线分为定向若干小段



② 和：在每一小段取点 (x_i, y_i) 求和 $\vec{F}(x_i,y_i)\cdot(\Delta x_i, \Delta y_i)$

· 为内积，向量 $(\Delta x_i, \Delta y_i)$ 为 L_i 所代表的方向向量。

求和可写为 $\vec{F}(x_i,y_i)\cdot(\Delta x_i,\Delta y_i) = P(x_i,y_i)\Delta x_i + Q(x_i,y_i)\Delta y_i$

③ 取极限：

$$\int_L \vec{F}\cdot d\vec{r} = \int_L P(x,y)\,dx + Q(x,y)\,dy$$

$$= \lim_{\lambda\to 0} \sum_i F(x_i,y_i)\cdot(\Delta x_i, \Delta y_i)$$

理解：变力做功。

区别：二型积分和定向有关!!!

第一型曲线积分计算公式

1. $y=y(x)$ on $[a,b]$

$$\int_L f(x,y)\,ds = \int_a^b f(x,y(x))\sqrt{1+(y'(x))^2}\,dx$$

直观解释：$|L_i| = \sqrt{(\Delta x_i)^2+(\Delta y_i)^2} = \sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$

$$\approx \sqrt{1+(y'(x_i))^2}\,\Delta x_i$$

2、L的方程为参数方程：

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad \alpha < t < \beta$$

$$\int_L f(x,y)\,ds = \int_\alpha^\beta f(x(t),y(t))\sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

推 1. & 2. !

Ex1: 考虑摆线 L: $x = t - \sin t \quad y = 1-\cos t \quad t \in (0,\pi]$

$$I = \int_L x\,ds$$

Pf: $ds = \sqrt{(1-\cos t)^2 + \sin^2 t}\;dt$

根据公式①:

$$I = \int_0^{2\pi} (t-\sin t)\underbrace{\sqrt{(1-\cos t)^2+\sin^2 t}}_{=\,2(1-\cos t)\,=\,4\sin^2\frac{t}{2}}\,dt$$

$$I = 2\int_0^{2\pi}(t-\sin t)\sin\frac{t}{2}\,dt$$

$$= 2\int_0^{2\pi}\left(t - 2\sin\frac{t}{2}\cos\frac{t}{2}\right)\sin\frac{t}{2}\,dt$$

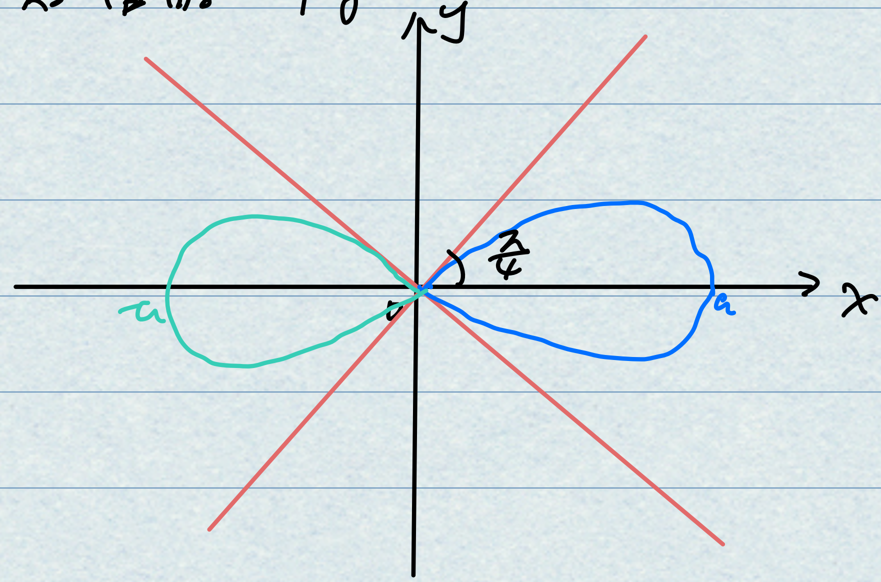
$$\left(u=\frac{t}{2}\right) \quad = 4\int_0^{\pi}(2u - 2\sin u\cos u)\sin u\,du$$

$$= 8\int_0^{\pi}(u-\sin u\cos u)\sin u\,du \quad \rightarrow \text{回顾} \int u\sin u\,du$$

$$= 8\left(\sin u - u\cos u + \frac{1}{3}\sin^3 u\right)\Big|_0^{\pi} = 8\pi. \qquad \square$$

Ex2:

求 $\oint_C (x+y)\,ds$ 其中 C 为双纽线 $r^2 = a^2\cos 2\theta$ 的右面的一瓣.



Pf:

Step 1: 想哪个公式可用 —— 参数方程!

$$\begin{cases} x = r(\theta)\cos\theta \\ y = r(\theta)\sin\theta \end{cases}$$

$$ds = \sqrt{r^2(\theta) + r'^2(\theta)}\,d\theta$$

$$r(\theta) = a\sqrt{\cos 2\theta} \qquad r'(\theta) = a\frac{(-2)\sin 2\theta}{2\sqrt{\cos 2\theta}}$$

$$\Rightarrow r(\theta)^2 + r'^2(\theta) = a^2\cos 2\theta + a^2\frac{(\sin 2\theta)^2}{\cos 2\theta}$$

$$= \frac{a^2}{\cos 2\theta}$$

$$\Rightarrow ds = a\frac{1}{\sqrt{\cos 2\theta}}\,d\theta$$

代入公式:

$$\int_C (x+y)\,ds = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r(\theta)(\sin\theta+\cos\theta)\, a\cdot\frac{1}{\sqrt{\cos 2\theta}}\,d\theta$$

$$= a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sin\theta+\cos\theta)\,d\theta$$

$$= a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\theta\,d\theta = a^2 \sin\theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \sqrt{2}a^2. \quad \square$$

第二型曲线积分的公式.

1. $y=y(x)$ $\quad a<x<b$

$$\int_L \vec{F}(x,y)\,d\vec{r} = \int_a^b \big(P(x,y(x)) + Q(x,y(x))y'(x)\big)\,dx$$

2. $\begin{cases} x=x(t) \\ y=y(t) \end{cases}$ $\quad \alpha<t<\beta$

$$\int_L \vec{F}(x,y)\,d\vec{r} = \int_\alpha^\beta \big(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\big)\,dt$$

Ex):

$$I = \int_C (x^2+2xy)\,dy$$

C为逆时针上半椭圆. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

pf: $\begin{cases} x=a\cos t \\ y=b\sin t \end{cases}$

$$I = \int_0^\pi (a^2\cos^2 t + 2ab\sin t\cos t)\, b\cos t\,dt$$

$$= \underbrace{a^2 b \int_0^{\pi} \cos^3 t \, dt}_{=0} + 2ab^2 \int_0^{\pi} \sin t \cos^2 t \, dt$$

$$= 2ab^2 \left(-\frac{\cos^3 t}{3} \right) \Big|_0^{\pi} = \frac{4}{3} a b^2$$

$\boxed{2}$

Ex 4: $\oint_C x^3 dy - y^3 dx$

C为圆周 $x^2+y^2=4$ 逆时针方向.

Pf:

$x = 2\cos t \quad y = 2\sin t \qquad 0 \le t \le 2\pi$

$$I = \oint_C x^3 dy - y^3 dx$$

$$= \int_0^{2\pi} 16\cos^3 t \cos t + 16 \sin^3 t \sin t \, dt$$

$$= 16 \int_0^{2\pi} \cos^4 t + \sin^4 t \, dt$$

$$= 32 \int_0^{2\pi} \sin^4 (t) \, dt$$

$$= 128 \int_0^{\frac{\pi}{2}} \sin^4 (t) \, dt$$

Recall:
$$\int_0^{\frac{\pi}{2}} \sin^n(\theta) \, d\theta = \begin{cases} \frac{(n-1)!!}{n!!} & n \text{ odd} \\ \frac{(n-1)!!}{n!!} \frac{\pi}{2} & n. \text{ even} \end{cases}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^4 (t) \, dt = \frac{3 \times 1}{4 \times 2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}$$

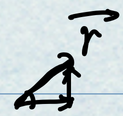
$$\Rightarrow I = 128 \times \frac{3\pi}{16} = 24\pi$$

∎

两类曲线积分的关系.

对于II型积分

$$I = \int_L \vec{F}(x,y) \cdot d\vec{r}.$$



注意到. $d\vec{r} = \vec{\tau} \cdot ds$

$$\Rightarrow I = \int_L \underbrace{\left(\vec{F}(x,y) \cdot \vec{\tau}\right)}_{\parallel \atop f(x,y)} ds$$

对于空间曲线 Γ. $\cos\alpha, \cos\beta, \cos\gamma$ 是切向余弦.

$$\int_\Gamma P\,dx + Q\,dy + R\,dz = \int_\Gamma (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,ds.$$

Ex5: 将

$$I = \int_L y\,dx + x\,dy + xyz\,dz \quad \text{化成一型}$$

$$L: \begin{cases} x = 2t \\ y = t^2 \\ z = t-1 \end{cases} \quad 0 \le t \le 1$$

Pf: $d\vec{r} = (x', y', z')\,dt = (2, 2t, 1)\,dt$

$$(\cos\alpha, \cos\beta, \cos\gamma) = \frac{d\vec{r}}{|d\vec{r}|} = \frac{(2, 2t, 1)}{\sqrt{4t^2+5}}$$

$$= \left(\frac{2}{\sqrt{4t^2+5}}, \frac{2t}{\sqrt{4t^2+5}}, \frac{1}{\sqrt{4t^2+5}} \right)$$

$$= \left(\frac{2}{\sqrt{4y+5}}, \frac{x}{\sqrt{4y+5}}, \frac{1}{\sqrt{4y+5}} \right)$$

$$I = \int_L \left(y \cdot \frac{2}{\sqrt{4y+5}} + x \cdot \frac{x}{\sqrt{4y+5}} + xyz \frac{1}{\sqrt{4y+5}} \right) ds$$

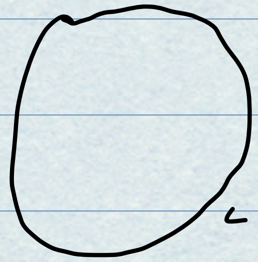
$$= \int_L \frac{2y+x^2+xyz}{\sqrt{5+4y}} \, ds.$$

格林公式，第二型曲线积分与路径无关条件.

回顾:

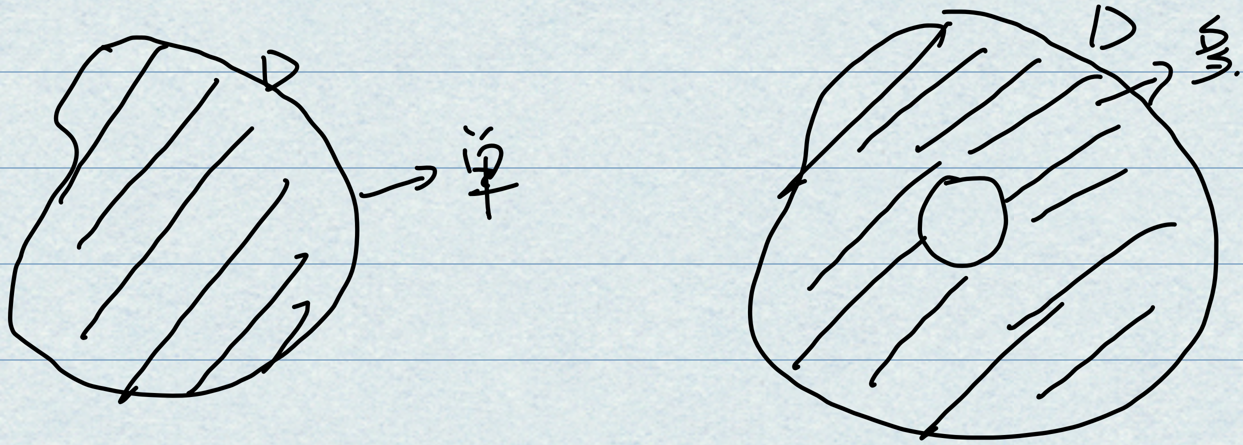
简单闭曲线 (Jordan 曲线): 是 $\varphi: [\alpha,\beta] \mapsto \mathbb{R}^2$ 的像,

φ 连续且为双射，$\varphi(\alpha) = \varphi(\beta)$

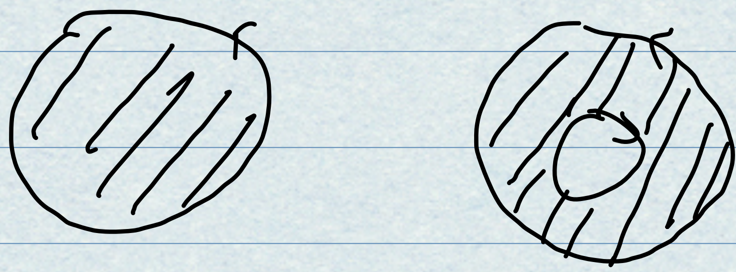


Jordan 引理: $\mathbb{R}^2 \setminus L = \Omega_1 \sqcup \Omega_2$，均为开集，且一个有界一个无界，有界称为内部.

定义1: 若 D 中任一条 Jordan curve 内部仍含于 D，则 D 单连通，otherwise 多连通.



曲线定向 L，L^+ (前进方向左手为内部)



格林公式 $P,Q\in C^1(D)$，L 逐段光滑.

$$\oint_{L^+} P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy.$$

阅读 例5 (P_{90}) 例6 (P_{92})

积分与路径无关条件. $P,Q\in C^1(D)$

$\int_{\widehat{AB}} P(x,y)\,dx+Q(x,y)\,dy$ 与路径无关.

$$\Updownarrow$$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

定理 设 $P,Q\in C^1(D)$，则

$$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}.$$

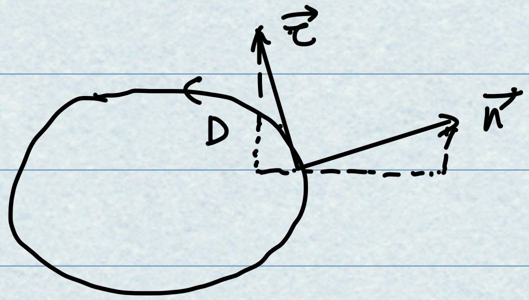
$$\Updownarrow$$

$\exists\, u(x,y)$. s.t. $du=P\,dx+Q\,dy$.

Ex1:

$$\oint_{L^+}\left[P\cos(\vec{n},x)+Q\cos(\vec{n},y)\right]ds=\iint_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)d\sigma$$

Pf:



$\vec{n}=(\cos(\vec{n},x),\cos(\vec{n},y))$

$\vec{\tau}=(-\cos(\vec{n},y),\cos(\vec{n},x))$

$$\text{LHS} = \oint_{L^+} (P, Q) \cdot (\cos(\vec{n}, x), \cos(\vec{n}, y))\, ds$$

$$= \oint_{L^+} (P, -Q)\, (\cos(\vec{n}, x), -\cos(\vec{n}, y))\, ds$$

$$= \oint_{L^+} (-Q, P) \cdot \vec{\tau}\, ds$$

$$= \oint_{L^+} -Q\, dx + P\, dy \overset{\text{Green}}{=} \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) d\sigma$$

<u>Ex2</u> 设 $u(x,y)$, $v(x,y)$ 有二阶连续偏导. 证明

(1) $$\iint_D v \Delta u \, d\sigma = \oint_{L^+} v \frac{\partial u}{\partial \vec{n}} ds - \iint_D \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) d\sigma$$

(2) $$\iint_D (u \Delta v - v \Delta u)\, d\sigma = \int_{L^+} u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}}\, ds$$

<u>Pf</u>: $$\oint_{L^+} v \cdot \frac{\partial u}{\partial \vec{n}}\, ds = \oint_{L^+} \left(v \cdot \frac{\partial u}{\partial x}, v \cdot \frac{\partial u}{\partial y}\right) \cdot \vec{n}\, ds$$

$$\overset{\text{Ex1}}{=} \iint_D \frac{\partial}{\partial x}\left(v \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v \frac{\partial u}{\partial y}\right) d\sigma$$

$$= \iint_D v \cdot \underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{\Delta u} + \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right) d\sigma$$

(2) follows by switching u and v.

$\boxtimes$

Ex3 设 $f(x,y)$ 在单位圆盘 $D=\{x^2+y^2\le 1\}$ 上 C^1，且有 $|f(x,y)|\le 1\ \forall (x,y)\in D$，证明：$\exists (x_0,y_0)\in \mathrm{int}\,D$, s.t.

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\Big|_{(x_0,y_0)}\le 4.$$

注：设 Γ 方程为

$x=x(t),\ y=y(t)$

满足 $\dfrac{dx}{dt}=\dfrac{1}{|\nabla f|}\dfrac{\partial f}{\partial x}$，$\dfrac{dy}{dt}=\dfrac{1}{|\nabla f|}\dfrac{\partial f}{\partial y}$

称 Γ 为 $f(x,y)$ 的梯度曲线

证：若 $\exists (x_0,y_0)\in D$. $\nabla f(x_0,y_0)=0$. 则证明完毕.

假设 $\nabla f\neq 0\ \forall (x,y)\in D$

考虑 f 在 D 中的梯度曲线 Γ:

初始 $\begin{cases} x(0)=0 \\ y(0)=0 \end{cases}$.

$$f(x(T),y(T))-f(0,0)=\int_0^T \frac{\partial f}{\partial x}x'(t)+\frac{\partial f}{\partial y}y'(t)\,dt$$

$$=\int_\Gamma \nabla f\cdot\vec{\tau}\,ds \qquad \vec{\tau}=\frac{\nabla f}{|\nabla f|}$$

$$|\mathrm{LHS}|\le 2$$

$$\Rightarrow \int_\Gamma |\nabla f|\,ds\le 2$$

$$\Rightarrow \exists (x_0,y_0)\ \text{s.t.}\ |\nabla f|\Big|_{(x_0,y_0)}\le 2 \qquad \square$$

第一型曲面积分

定义:

分: $S \mapsto \Delta S_i$

取点: $(\xi_i, \eta_i, \zeta_i) \in \Delta S_i$

求和: $\sum_i \rho(\xi_i, \eta_i, \zeta_i)|\Delta S_i|$

取极限: $\lambda = \max_i |\Delta S_i|$

$$\lim_{\lambda \to 0} \sum_i \rho(\xi_i, \eta_i, \zeta_i)|\Delta S_i|$$

记为 $\iint_S \rho(x, y, z)\, dS$

公式:

(1) $z = g(x, y) \quad (x, y) \in D$

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{1 + z_x^2 + z_y^2}\, d\sigma$$

(2) $\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} \quad (u, v) \in D$

$$\iint_S f(x, y, z)\, dS = \iint_D f(x(u,v), y(u,v), z(u,v)) \sqrt{EG - F^2}\, du\, dv$$

where $\begin{cases} E = x_u^2 + y_u^2 + z_u^2 \\ F = x_u x_v + y_u y_v + z_u z_v \\ G = x_v^2 + y_v^2 + z_v^2 \end{cases}$

Ex 4

$I = \oiint_S (xy+z)^2 dS \qquad S: x^2+y^2+z^2=R^2$

pf: $I = \oiint_S (x^2y^2+z^2) dS$

$\begin{cases} z = R\cos\varphi \\ x = R\sin\varphi\cos\theta \\ y = R\sin\varphi\sin\theta \end{cases} \qquad \begin{cases} E = R^2\sin^2\varphi \\ G = R^2\sin^2\varphi + R^2\cos^2\varphi\cos^2\theta + R^2\cos^2\varphi\sin^2\theta = R^2 \\ F = 0 \end{cases}$

$\Rightarrow I = \int_0^{2\pi}\int_0^{\pi} (R^4\sin^4\varphi\sin^2\theta\cos^2\theta + R^2\cos^2\varphi) R^2\sin\varphi \, d\varphi \, d\theta$

$= R^6 \int_0^{2\pi} \sin^2\theta\cos^2\theta \, d\theta \cdot \int_0^{\pi} \sin^5\varphi \, d\varphi + 2\pi R^4 \int_0^{\pi} \cos^2\varphi\sin\varphi \, d\varphi$

$= R^6 \cdot \frac{\pi}{4} \cdot 2 \cdot \frac{4\times 2}{5\cdot 3} + 2\pi R^4 \cdot \frac{2}{3}$

$= \frac{4\pi}{15} R^6 + \frac{4\pi}{3} R^4$

第二型曲面积分

双侧曲面.(莫比乌斯带)

$f(x,y,z)=0, \quad \vec{n} = (f_x, f_y, f_z)$

定义:

分: ΔS_i

求和: $\sum_{i=1}^{n} F(\xi_i, \eta_i, \zeta_i) \vec{n}(\xi_i, \eta_i, \zeta_i) \Delta S_i$

极限: $\lim_{\lambda\to 0} \sum_{i=1}^{n} F(\cdots) \vec{n}(\cdots) \Delta S_i$

$$\iint_S \vec{F} \cdot (dydz, dxdz, dxdy) = \iint_S F(x,y,z) \cdot \vec{n}(x,y,z)\, dS$$

公式. 若 $x = x(y,z)$

$$\iint_S P\, dydz + Q\, dzdx + R\, dxdy$$

$$= \pm \iint_{D_{yz}} P(x(y,z),y,z) + Q(x(y,z),y,z)(-x_y)$$

$$+ R(x(y,z),y,z)(-x_z)\, dydz$$

<u>Ex 5</u>

$$\iint_S z\, dxdy \qquad S \text{ 为 } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ 椭外侧.}$$

pf: $S = S_1 \cup S_2$

$$I = 2\iint_{S_1} z\, dxdy \quad , \quad z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$= 2 \iint_D c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dxdy$$

$$= 2 \cdot \frac{2}{3}\pi abc = \frac{4}{3}\pi abc .$$

$\boxed{3}$

Caveat

Ex6 若 Σ 分块光滑，且关于 xoy 对称，$f(x,y,z)$ 在 Σ 上连续，且 $f(x,y,z)=-f(x,y,-z)$，$\Sigma=\Sigma_1\cup\Sigma_2$

Σ_1 上 $\quad$ Σ_2 下

① $\iint_\Sigma f(x,y,z)dS \overset{?}{=} 2\iint_{\Sigma_1} f(x,y,z)dS$

② $\iint_\Sigma f(x,y,z)\,dxdy \overset{?}{=} 2\iint_{\Sigma_1} f(x,y,z)\,dxdy$.

公式 2.

$$S:\begin{cases} x=x(u,v) \\ y=y(u,v) & (u,v)\in D' \\ z=z(u,v) \end{cases}$$

$$A=\frac{D(y,z)}{D(u,v)} \qquad B=\frac{D(z,x)}{D(u,v)} \qquad C=\frac{D(x,y)}{D(u,v)}$$

$$\iint_S P\,dydz + Q\,dzdx + R\,dxdy$$

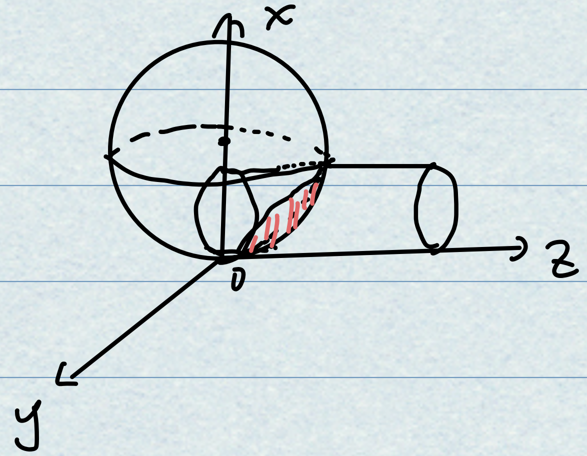
$$=\pm\iint_{D'}\left[P(x,y,z)A(u,v)+Q(x,y,z)B(u,v)+R(x,y,z)C(u,v)\right]dudv$$

Ex7 Calculate

$$I=\iint_S (y-z)\,dydz+(z-x)\,dxdz+(x-y)\,dxdy$$

S：为 $(x-R)^2+y^2+z^2=R^2$ 被 $(x-r)^2+y^2=r^2$ 割下部分在 $z\geq 0$ 部分 外侧 $R>r$

pf:



$z = \sqrt{2Rx - x^2 - y^2}$ $\quad D: x^2+y^2 \le 2rx$

单位法向.

$$\vec{n}(x,y,z) = \pm \frac{1}{\sqrt{1+\frac{y^2+(R-x)^2}{z^2}}}\left(-\frac{R-x}{z}, \frac{y}{z}, 1\right)$$

取"+".

$$I = \iint_S (y-z, z-x, x-y)\cdot \vec{n}\, ds$$

$$= \iint_S \frac{1}{\sqrt{1+\frac{y^2+(R-x)^2}{z^2}}}\left(\frac{x-R}{z}, \frac{y}{z}, 1\right)\cdot(y-z, z-x, x-y)\, ds$$

$$= \iint_D \frac{(x-R)(y-z)}{z} + \frac{y(z-x)}{z} + (x-y)\, dxdy$$

$$= \iint_D \frac{xy - xz - Ry + Rz + yz - yx + xz - yz}{z}\, dxdy$$

$$= \iint_D \left(R - R\cdot\frac{y}{z}\right) dxdy$$

$$= R\iint_D dxdy = \pi r^2 \cdot R$$ $\square$

Ex8 (bonus)

设 $u(x,y)$ 在 D 上调和，即 $u \in C^2(D)$ 且

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on } D$$

证明：

(1) $\oint_{C^+} u \frac{\partial u}{\partial n} ds = \iint_D \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 d\sigma$

(2) 若 $u(x,y)$ 在 L 上为零，则 $u(x,y)$ 在 D 上为零。

Ex9 (Mean-value theorem)

$u \in C^2(D)$ harmonic. $\Delta u = 0$ then

$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dy$$

pf: 令 $E(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dy.$

Let $y = x + rz \quad z \in \partial B(0,1)$

$$E(r) = \frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} u(x+rz)\, dz$$

$$E'(r) = \frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} z \cdot \nabla u(x+rz)\, dz$$

$$= \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \frac{y-x}{r} \cdot \nabla u(y)\, dy$$

$$= \frac{1}{|\partial B(x,r)|} \underbrace{\int_{\partial B(x,r)} \frac{\partial u}{\partial \vec{n}}\, dy}_{\int_{B(x,r)} \Delta u\, dy} = 0$$

$\Rightarrow E(r) = c$ for some c,

Let $r \to 0$. $E(r) \to u(x)$. $\square$

Gauss 公式

设S是有界空间Ω的边界，那么 Gauss 公式的形式是

$$\iint_{S^+} P\,dydz + Q\,dzdx + R\,dxdy$$

$$= \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz$$

S^+：曲面外侧.

散度 (divergence)：向量函数 $\vec{F} = (P, Q, R)$

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Gauss 公式有如下写法：

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_{\Omega} \operatorname{div} \vec{F}\, dV.$$

可推广至 n 维：$\Omega \in \mathbb{R}^n$ 有界区域 $\vec{F} = (P_1 \cdots P_n)$

$$\int_{\partial\Omega} (\vec{F} \cdot \vec{n})\, dS = \iiint_{\Omega} \operatorname{div} \vec{F}\, d\vec{x}$$

where. $\operatorname{div} \vec{F} = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i}$

Stokes 公式.

S 双侧有界曲面，L为 S 的边界，那么

$$\oint_L P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx$$

$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

右手系:

$$= \iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Ex1: In Stoke's formula. If L is the boundary of both S_1 and S_2. We have

$$\iint_{S_1} \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \iint_{S_2} \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Explain why?

Ex2. S 光滑闭曲面. 围成 Ω. $u(x,y,z), v(x,y,z) \in C^2(\Omega \cup S)$

证明:

① $\iiint_\Omega \frac{\partial u}{\partial x} v \, dV = -\iiint_\Omega u \cdot \frac{\partial v}{\partial x} dV + \iint_S u v \, n_1 \, dS.$

其中 $\vec{n} = (n_1, n_2, n_3)$.

② $\iiint_\Omega \nabla u \cdot \nabla v \, dV = -\iiint_\Omega u \Delta v \, dV + \iiint_S u \cdot \frac{\partial v}{\partial n} dS.$

③ If $\Delta u = 0$ and $u(x,y,z) \equiv 0$ on S, then show that $u(x,y,z) \equiv 0$ on Ω..

常微分方程 (ODE)

Background:

(i) Newton's second law of motion.

$$\begin{cases} \dfrac{d^2x}{dt^2} = a \\ F = ma \end{cases} \Rightarrow F = m \cdot \frac{d^2x}{dt^2}$$

(ii) The mass of radium.

$$\frac{dm(t)}{dt} = -\rho m(t)$$

n阶微分方程一般形式

$$F(x, y(x), y'(x), \cdots, y^{(n)}(x)) = 0$$

通解，特解

ode 解法：

① 分离变量

(i) $y' = f(x)g(y) \Leftrightarrow \frac{1}{g(y)} dy = f(x) dx$

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

(ii) 齐次方程

$$y' = f(x, y) = g\left(\frac{y}{x}\right)$$

令 $u=\frac{y}{x}$ $\quad u'=\frac{xy'-y}{x^2}=\frac{xg(u)-ux}{x^2}=\frac{g(u)-u}{x}$

$\Downarrow$

分离变量.

(iii) $y'=f(ax+by+c)$

令 $z=ax+by+c$

$z'=a+by'=a+bf(z)$

(iv) $y'=f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$

① 若 $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0$ 令 $z=a_1x+b_1y$.

② $c_1=c_2=0 \Rightarrow$ 齐次

③ $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\neq 0$

Solve $\begin{cases} a_1x_0+b_1y_0+c_1=0 \\ a_2x_0+b_2y_0+c_2=0 \end{cases} \Rightarrow (x_0, y_0)$

$\Rightarrow y'=f\left(\frac{a_1(x-x_0)+b_1(y-y_0)}{a_2(x-x_0)+b_2(y-y_0)}\right)$ $\quad \begin{cases} u=x-x_0 \\ v=y-y_0 \end{cases}$

$\Rightarrow$ 齐次

Ex3. $(x^2+1)(y^2-1)dx+xydy=0$

pf: $\frac{x^2+1}{x}dx=-\frac{y}{y^2-1}dy$

$$\Rightarrow \int \frac{x^2+1}{x}\,dx = \int -\frac{y}{y^2-1}\,dy$$

$$\Rightarrow \frac{x^2}{2} + \log|x| = -\frac{1}{2}\log|y^2-1| + c$$

$$\Rightarrow e^{x^2}\cdot x^2\,|y^2-1| = e^{c}$$

$$\Rightarrow e^{x^2}x^2\cdot(y^2-1) = c_1$$

Ex 4. $y^2x+3+3y^2+x = y'$

pf: $(y^2+1)(x+3) = y'$

$$\frac{1}{y^2+1}\,dy = (x+3)\,dx$$

$$\arctan(y) = \log|x+3| + c$$

$$y = \tan(\log|x+3| + c)$$

Ex 5. $y' = (8x+2y+1)^2$

pf: $z = 8x+2y+1$

$$z' = 8+2y' = 8+2z^2$$

$$\Rightarrow \frac{1}{8+2z^2}\,dz = dx$$

$$\frac{1}{4}\arctan\left(\frac{z}{2}\right) = x + C$$

代入 $\arctan(4x+y+\frac{1}{2}) = 4x + C.$

一阶线性方程.

$$y' + P(x)y = 0$$

$$\Rightarrow y = C\exp\left(-\int P(x)dx\right)$$

非齐次一阶线性方程.

$$y' + P(x)y = Q(x) \quad (*)$$

方法: 常数变易法.

先解 $y' + P(x)y = 0$

$$\Rightarrow y = C(x)\exp\left(-\int p(x)dx\right)$$

代入 $(*)$ 求 $C(x)$

$$C'(x)\exp\left(-\int p(x)dx\right) = Q(x)$$

解 $C(x)$.

Bernoulli 方程.

$$y' + P(x)y = Q(x)y^{\alpha}$$

令 $z = y^{1-\alpha}$

Ex 6. $y' + y = y^2(\cos x - \sin x)$

pf: 令 $z = y^{-1}$

$$z' - z = \sin x - \cos x$$

$$z' - z = 0 \Rightarrow z = ce^{x}$$

设 $z = c(x)e^{x}$

$$z' - z = c'(x)e^{x} = \sin x - \cos x$$

$$c'(x) = e^{-x}(\sin x - \cos x)$$

$$\Rightarrow c(x) = \int (\sin x - \cos x)\, d(-e^{-x})$$

$$= -e^{-x}(\sin x - \cos x) + \int e^{-x}(\cos x + \sin x)\, dx$$

$$= -e^{-x}(\sin x - \cos x) - \int (\cos x + \sin x)\, d(e^{-x})$$

$$= -e^{-x}(\sin x - \cos x) - e^{-x}(\cos x + \sin x) + \int e^{-x}(\cos x - \sin x)\, dx$$

$$\Rightarrow c(x) = -e^{-x}\sin x + C$$

$$\Rightarrow z = (-e^{-x}\sin x + c)e^{x} = -\sin x + ce^{x}$$

$$y = \frac{1}{ce^{x} - \sin x}$$

$\boxed{3}$

Ex 7. $\dfrac{dy}{dx} + 2y = 4x$

pf: $y = c(x)e^{-2x}$

$$c'(x)e^{-2x} = 4x$$

$$c'(x) = 4xe^{2x}$$

$$c(x) = \int 4xe^{2x}\, dx = \int 2x\, d(e^{2x})$$

$$= 2x e^{-2x} - \int e^{2x} d(2x) = (2x-1) e^{2x} + C$$

$$\Rightarrow y = (2x-1) + Ce^{-2x}$$

恰当方程:

形式: $P(x,y)dx + Q(x,y)dy = 0$

$$\partial_y P = \partial_x Q$$

找 $u(x,y)$ s.t. $du = Pdx + Qdy$

则 $u(x,y) = C$

积分因子法:

$$M(x,y)dx + N(x,y)dy = 0$$

找 $\mu(x,y)$, s.t.

$\mu M dx + \mu N dy = 0$ 为恰当.

理解: $\frac{\partial \mu}{\partial y} \cdot M + \frac{\partial M}{\partial y} \mu = \frac{\partial \mu}{\partial x} N + \frac{\partial N}{\partial x} \mu$

$$\frac{1}{\mu}\left(\frac{\partial \mu}{\partial y} M - \frac{\partial \mu}{\partial x} N\right) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

若 $\mu = \mu(x)$, 则

$$\frac{1}{\mu} \mu'(x) = \boxed{\frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)} =: F(x)$$

则 $\mu(x) = \exp\left(\int F(x)\, dx\right)$ ✓

② $u=u(y)$ 同理.

Ex 8. $y\,dx+(x-1-xy)dy=0$

pf: 设. $M=y$ $N=x-1-xy$.

$uM dx+uN dy=0$

若恰当 则

$$\frac{\partial u}{\partial y}M+u=\frac{\partial u}{\partial x}N+u\cdot(1-y)$$

$$\Rightarrow \frac{\partial u}{\partial y}y=\frac{\partial u}{\partial x}(x-1-xy)-u\cdot y$$

设 $\frac{\partial u}{\partial x}=0$, 则

$$\frac{\partial u}{\partial y}=-u \Rightarrow u=e^{-x}$$

$$\Rightarrow ye^{-y}dx+e^{-y}(x-1-xy)dy=0 \text{ 恰当}$$

$\Downarrow \rightarrow$ 求原函数.

$$(xy+1)e^{-y}=c$$

$\square$

Ex1 $y'' + q(x)y = 0 \quad (*)$

证 $\forall\ \varphi_1, \varphi_2$ 为 $(*)$ 的解，有 $W(x) \equiv$ 常数.

Pf: $W(x) = \varphi_1(x)\varphi_2'(x) - \varphi_2(x)\varphi_1'(x)$

$$W'(x) = \varphi_1'(x)\varphi_2'(x) + \varphi_2''(x)\varphi_1(x) - \varphi_2'(x)\varphi_1'(x) - \varphi_2(x)\varphi_1''(x)$$

$$= -q(x)\varphi_2(x)\varphi_1(x) + \varphi_2(x)q(x)\varphi_1(x) \equiv 0.$$

Ex2 证明：

$y''(x) + p(x)y'(x) + q(x)y(x) = 0$ 存在两个线性无关的解.

证明：考虑初值问题

$$\begin{cases} y''(x) + p(x)y' + q(x)y = 0 \\ y(x_0) = \alpha_0,\ y'(x_0) = \alpha_1 \end{cases} \quad (*)$$

and

$$\begin{cases} y''(x) + p(x)y' + q(x)y = 0 \\ y(x_0) = \beta_0,\ y'(x_0) = \beta_1 \end{cases} \quad (**)$$

则 (1)、(2) 均有唯一解，φ_1 与 φ_2.

$$W(x)=\begin{vmatrix}\varphi_1(x) & \varphi_2(x)\\ \varphi_1'(x) & \varphi_2'(x)\end{vmatrix}$$

则 $W(x_0)=\begin{vmatrix}\alpha_0 & \beta_0\\ \alpha_1 & \beta_1\end{vmatrix}\neq 0$

则 φ_1 φ_2 线性无关

Ex3 考虑上一方程. 证明任两线性无关的解无公共零点.

Pf: $W(x)=\begin{vmatrix}\varphi_1(x) & \varphi_2(x)\\ \varphi_1'(x) & \varphi_2'(x)\end{vmatrix}$

若有 x_0 为公共零点. 则 $W(x_0)=0$，与线性无关矛盾.

Ex4 设 $\varphi(x)$ 为

(*) $y''(x)+p(x)y'(x)+q(x)y(x)=0$ 的解

若有 $x_0\in(a,b)$ $\varphi(x_0)=0$. 证明 $\varphi'(x_0)\neq 0$

Pf: 且 $\tilde\varphi$ 与 φ 线性无关. 则

$$W(x)=\begin{vmatrix}\tilde\varphi & \varphi\\ \tilde\varphi'(x) & \varphi'(x)\end{vmatrix}\neq 0\quad \forall x$$

若 $\exists x_0$. $\varphi(x_0)=\varphi'(x_0)=0$, 则 $W(x_0)=0$. 矛盾.